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BAYES ESTIMATES OF THE VARIANCE OF A NORMAL POPULATION FOR PRIO--ETC(U)
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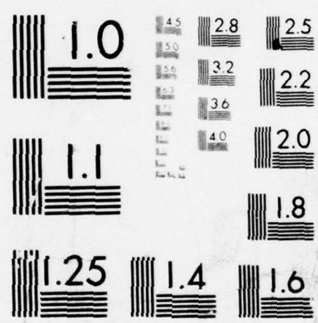

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BAYES ESTIMATES OF THE VARIANCE OF A NORMAL POPULATION FOR PRIOR CONJUGATE DISTRIBUTIONS OF INDEPENDENT PARAMETERS WITH APPLICATION TO ESTIMATION IN FINITE POPULATIONS,

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Bayes Estimates of the Variance of a Normal Population for Prior Conjugate Distributions of Independent Parameters with Application to Estimation in Finite Populations

by

S. Zacks

1. Introduction

Let $X_1,\ldots,X_n$ be i.i.d. random variables having a normal distribution $N(\mu,\sigma^2)$, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$, where both μ and σ are unknown. Consider the problem of estimating the distribution variance σ^2. Let $\bar{X}_n$, S_n^2 be the sample mean and the sample variance, respectively. S_n^2 is an equivariant estimator, with respect to the group $\mathcal{G}$ of real affine transformations. It is well known that S_n^2 is inadmissible for the squared-error loss function. Moreover, all equivariant estimators of σ^2 are inadmissible (see Zacks [3; pp. 364]), this is due to the fact that $\bar{X}_n$ and S_n^2 are independent and $\bar{X}_n$ has also some information on σ^2, that can be utilized to reduce the mean-squared-error (MSE) of the variance estimator. Bayes estimators of σ^2, with respect to the squared-error loss, for any prior distributions having positive p.d.f. for all points in the parameter space, are admissible estimators (see Zacks [3; pp. 365]). The question is whether such admissible Bayes estimators are substantially more efficient than the minimum-MSE equivariant estimator $\hat{\sigma}_E^2 = \frac{n-1}{n+1} S_n^2$. In the normal case the proper Bayes estimators of σ^2 have more complicated form than $\hat{\sigma}_E^2$ and sometimes a computer is needed for their application. However, today the need for using a computer is not an obstacle. The justification for using a complicated estimator is only in substantial improvement of efficiency. Box and Tiao [1] and Zellner [5] provide formulae of formal Bayes estimators of σ^2, using the improper Jeffery's prior $H(\mu,\sigma) = d\mu d\sigma/\sigma$.

The corresponding formal Bayes estimator is however equivariant and is not better than $\hat{\sigma}_E^2$. DeGroot [2] presents a proper Bayes estimator which is admissible. This Bayes estimator is derived for the prior conjugate distributions so that, given σ^2, the conditional prior distribution of μ is normal $N(\bar{\mu}, \kappa^2\sigma^2)$ and the prior marginal distribution of $1/2\sigma^2$ is the gamma distribution $\text{Gamma}(\psi, \nu)$, where ψ^{-1} is the scale parameter. The Bayes estimator corresponding to this prior model is

$$\hat{\sigma}_{BD}^2 = \frac{\psi + (n-1)S_n^2 + n(\bar{X}_n - \bar{\mu})^2/(1+n\kappa^2)}{n + 2\nu - 2} . \tag{1.1}$$

In the present paper we develop the Bayes estimator suggested by Zacks [3; pp. 366], assuming conjugate distributions of independent prior parameters μ and σ^2. More specifically, we assume that $\mu \sim N(\bar{\mu}, D^2)$ and $1/2\sigma^2 \sim \text{Gamma}(\psi, \nu)$. Although the difference between DeGroot's model and the present one seems small, the two models are actually quite different, since in the former the prior normal distribution of μ depends on σ^2. In the present model of priorly independent parameters we obtain a substantially more complicated Bayes estimator, designated by $\hat{\sigma}_{BI}^2$. In Section 3 we compare the relative efficiencies of the three estimators $\hat{\sigma}_E^2$, $\hat{\sigma}_{BD}^2$ and $\hat{\sigma}_{BI}^2$. The Bayesian framework developed here is applied in Section 4 for the derivation of the Bayes estimator of the variance σ_N^2 of a finite population, which is discussed by Zacks and Solomon [4].

2. Derivation of the Bayes Estimator $\hat{\sigma}_{BI}^2$

Let $\theta = 1/2\sigma^2$. The likelihood function of (μ, θ) given the minimal sufficient statistic $(\bar{X}_n, S_n^2)$ is

$$L(\mu,\theta|\bar{X}_n,S_n^2) = \theta^{\frac{n}{2}} \exp\{-n\theta(\bar{X}_n-\mu)^2 - (n-1)\theta S_n^2\}, \tag{2.1}$$

for $-\infty < \mu < \infty$ and $0 < \theta < \infty$. It follows that the posterior p.d.f of (μ,θ), given $(\bar{X}_n, S_n^2)$, for the independent conjugate priors is

$$k(\mu,\theta|\bar{X}_n,S_n^2) \propto \theta^{\frac{n}{2}+\nu-1} \exp\{-n\theta(\bar{X}_n-\mu)^2 - \frac{1}{2D^2}(\mu-\bar{\mu})^2 - \theta[(n-1)S_n^2+\psi]\}, \quad -\infty < \mu < \infty, \quad 0 < \theta < \infty. \tag{2.2}$$

It is easy to verify that

$$\int_{-\infty}^{\infty} \exp\{-n\theta(\bar{X}_n-\mu)^2 - \frac{1}{2D^2}(\mu-\bar{\mu})^2\}d\mu = \sqrt{2\pi}\, D(1+2n\theta D^2)^{-\frac{1}{2}} \exp\{-\frac{n\theta}{1+2n\theta D^2}(\bar{X}_n-\bar{\mu})^2\}. \tag{2.3}$$

Hence, the posterior expectation of $\sigma^2 = 1/2\theta$, given $(\bar{X}_n, S_n^2)$, is

$$E\{\sigma^2|\bar{X}_n,S_n^2) = \frac{1}{2}\cdot\frac{\int_0^{\infty}\theta^{\frac{n}{2}+\nu-2}(1+2n\theta D^2)^{-\frac{1}{2}}\exp\left\{-\frac{n\theta}{1+2n\theta D^2}(\bar{X}_n-\bar{\mu})^2-\theta((n-1)S_n^2+\psi)\right\}d\theta}{\int_0^{\infty}\theta^{\frac{n}{2}+\nu-1}(1+2n\theta D^2)^{-\frac{1}{2}}\exp\left\{-\frac{n\theta}{1+2n\theta D^2}(\bar{X}_n-\bar{\mu})^2-\theta((n-1)S_n^2+\psi)\right\}d\theta}. \tag{2.4}$$

This is the Bayes estimator $\hat{\sigma}_{BI}^2$. By making the transformation $X = 2n\theta D^2$ we reduce (2.4) to

$$\hat{\sigma}_{BI}^2 = \frac{(n-1)S_n^2+\psi}{n+2\nu-2}\cdot\frac{M_1(\lambda,\frac{n}{2}+\nu-1,\delta^2)}{M_1(\lambda,\frac{n}{2}+\nu,\delta^2)}, \tag{2.5}$$

where

$$(2.6)\qquad \lambda = ((n-1)S_n^2 + \psi)/2nD^2, \quad \delta^2 = (\bar{X}_n - \bar{\mu})^2/D^2$$

and for each $r = 1,2,\ldots$ and $X \sim \text{Gamma}(\lambda,\nu)$

$$(2.7)\qquad M_r(\lambda,\nu,\delta^2) = E\{(1+X)^{-\frac{r}{2}} \exp\{-\frac{\delta^2}{2}\cdot\frac{X}{1+X}\}\}.$$

The function $M_r(\lambda,\nu,\delta^2)$ is determined in the following manner. We make first the expansion

$$(2.8)\qquad M_r(\lambda,\nu,\delta^2) = \sum_{j=0}^{\infty}\left(-\frac{\delta^2}{2}\right)^2 \frac{1}{j!} E_{\lambda,\nu}\left\{\frac{X^j}{(1+X)^{j+r/2}}\right\}.$$

Let $p(j|\lambda)$ denote the p.d.f. of the Poisson with mean λ. Then,

$$(2.9)\qquad M_r(\lambda,\nu,\delta^2) = \sum_{j=0}^{\infty} p(j|\frac{\delta^2}{2})R_j(r,\lambda,\nu,\delta^2),$$

where

$$(2.10)\qquad R_j(r,\lambda,\nu,\delta^2) = (-1)^j \frac{\lambda^\nu e^{\delta^2/2}}{\Gamma(\nu)} \int_0^\infty \frac{x^{j+\nu-1}}{(1+x)^{j+r/2}} e^{-\lambda x}\,dx$$

$$= (-1)^j \frac{\lambda^\nu e^{\delta^2/2+\lambda}}{\Gamma(\nu)} \int_1^\infty \frac{(y-1)^{j+\nu-1}}{y^{j+r/2}} e^{-\lambda y}\,dy \quad .$$

Suppose that ν is an integer and $r = 2m+1$, then

$$(2.11)\qquad R_j(r,\lambda,\nu,\delta^2) = (-1)^j \frac{\lambda^\nu e^{\delta^2/2+\lambda}}{\Gamma(\nu)} \sum_{i=0}^{j+\nu-1} (-1)^i \binom{j+\nu-1}{i} E_{\nu-1-m-\frac{1}{2}}(\lambda)$$

where generally

$$(2.12)\qquad E_{\ell-\frac{1}{2}}(\lambda) = \int_1^\infty y^{\ell-\frac{1}{2}} e^{-\lambda y}dy, \qquad \ell = 0,\pm1,\pm2,\ldots\ .$$

These exponential integrals are determined by the recursive formula

$$(2.13)\qquad E_{\ell-\frac{1}{2}}(\lambda) = \begin{cases} (e^{-\lambda} + (\ell-\frac{1}{2})E_{\ell-\frac{3}{2}}(\lambda))/\lambda\,, & \ell \ge 1 \\ \frac{1}{-\ell-\frac{1}{2}}(e^{-\lambda} - \lambda\, E_{-\ell+\frac{1}{2}}(\lambda))\,, & \ell \le -1 \end{cases}$$

where

$$(2.14)\qquad E_{-\frac{1}{2}}(\lambda) = 2\sqrt{\frac{\pi}{\lambda}}\,(1 - \Phi(\sqrt{2\lambda}))\,,$$

and $\Phi(z)$ is the standard normal integral. If ν is not an integer other expansions can be attempted or the value of the M-function can be determined approximately by linear interpolation between the values of the M-function corresponding to the two integers adjacent to ν. In the Appendix we provide a FORTRAN subroutine function to compute $M_r(\lambda,\nu,\delta)$ for integer values of ν.

3. Relative Efficiency Comparisons

In the present section we compare the three estimators $\hat{\sigma}_E^2$, $\hat{\sigma}_{BD}^2$ and $\hat{\sigma}_{BI}^2$ with respect to their relative efficiency. For the purpose of comparing the Bayes estimators with the equivariant ones we define the relative efficiency of an estimator $\hat{\sigma}^2$ as the ratio of its MSE to that of $\hat{\sigma}_E^2$. More specifically, the relative efficiency function of $\hat{\sigma}^2$ is defined as

$$(3.1)\qquad RE(\hat{\sigma}^2,\omega) = \frac{n+1}{2\sigma^4} / E_2\{(\hat{\sigma} - \sigma^2)^2\},$$

where $\omega = (\mu,\sigma^2)$. We derive first the relative efficiency of $\hat{\sigma}_{BD}^2$. Notice that $n(\bar{X}_n-\bar{\mu})^2 \sim \sigma^2\chi^2[1;\ \frac{n(\mu-\bar{\mu})^2}{2\sigma^2}]$, where $\chi^2[1;\lambda]$ designates the non-central chi-squared with 1 degree of freedom and parameter of non-centrality λ. Let $n' = n+2\nu-2$ then

$$(3.2)\qquad \hat{\sigma}^2_{BD} \sim \frac{\psi}{n'} + \frac{\sigma^2\chi^2_1[n-1]}{n'} + \frac{\sigma^2\chi^2_2[1;\frac{n(\mu-\bar{\mu})^2}{2\sigma^2}]}{n'(1+n\varkappa^2)} ,$$

with $\chi^2_1[\cdot]$ and $\chi^2_2[\cdot;\cdot]$ independent. Hence,

$$(3.3)\qquad E\{\hat{\sigma}^2_{BD}\} = \sigma^2 + \frac{\sigma^2}{n'}\left\{\frac{1}{1+n\varkappa^2}\left(1+\frac{n(\mu-\bar{\mu})^2}{\sigma^2}\right) - (2\nu-1) + \frac{\psi}{\sigma^2}\right\}$$

and

$$(3.4)\qquad \mathrm{Var}\{\hat{\sigma}^2_{BD}\} = \frac{2\sigma^4}{(n')^2}\left[n-1+\frac{1+4n\left(\frac{\mu-\bar{\mu}}{\sigma}\right)^2}{(1+n\varkappa^2)^2}\right] .$$

Let $\zeta = (\mu-\bar{\mu})/\sigma$ and $\beta = \frac{1+n\zeta^2}{1+n\varkappa^2} - (2\nu-1) + \frac{\psi}{\sigma^2}$ then the relative efficiency of $\hat{\sigma}^2_{BD}$ depends only on ζ^2, $\frac{\psi}{\sigma^2}$, $\varkappa^2$, ν and n and is given by:

$$(3.6)\qquad RE(\hat{\sigma}^2_{BD};\zeta^2,\frac{\psi}{\sigma^2},r,\nu) = \frac{n+2\nu-2}{n+1}\left[1-\frac{2\nu-1}{n+2\nu-2} + \frac{1+4n\zeta^2}{(n+2\nu-2)(1+n\varkappa^2)} + \frac{\beta^2}{2(n+2\nu-2)}\right]^{-1} .$$

The estimator $\hat{\sigma}^2_{BI}$ is considerably more complicated and no explicit formula of its MSE can be derived. We can compute its MSE, however, numerically in the following manner. Since $n(\bar{X}_n-\bar{\mu})^2 \sim \sigma^2\chi^2[1;\lambda]$ with $\lambda = \frac{n}{2}\zeta^2$ we can write

$$(3.7)\qquad E\{(\hat{\sigma}^2_{BI}-\sigma^2)^2\} = \sigma^4 E\left\{\left[\frac{\frac{\psi}{\sigma^2}+2W_1}{n'} \cdot \frac{M_1\left(\left(\frac{\psi}{\sigma^2}+2W_1\right)/2n\varkappa^2, \frac{n}{2}+\nu-1, 2W_2(J)/n\varkappa^2\right)}{M_1\left(\left(\frac{\psi}{\sigma^2}+2W_1\right)/2n\varkappa^2, \frac{n}{2}+\nu, 2W_2(J)/n\varkappa^2\right)} - 1\right]^2\right\},$$

where $W_1, W_2(J)$ are independent, $W_1 \sim \mathrm{Gamma}\left(1,\frac{n-1}{2}\right)$, $W_2(J) \sim \mathrm{Gamma}(1,\frac{1}{2}+J)$ and J is a Poisson r.v. with mean λ. Let $G(x|\nu)$ be the c.d.f of

Gamma$(1,\nu)$ at x, let $G^{-1}(p|\nu)$ be the p-th fractile of Gamma $(1,\nu)$. Define $\bar{\xi}_1 = G^{-1}(.99|\frac{n-1}{2})$, $\underline{\xi}_2(j) = G^{-1}(.005|\frac{1}{2}+j)$ and $\bar{\xi}_2(j) = G^{-1}(.995|\frac{1}{2}+j)$. The risk function (3.7) is determined by computing first the conditional expectation given J numerically over the range $(0,\bar{\xi}_1)\times(\underline{\xi}_2(J),\bar{\xi}_2(J))$. The conditional expectations are then averaged with respect to the Poisson distribution with mean λ. The range of integration for each J is partitioned into $M\times M$ rectangles. Let $\xi_1(i) = i\bar{\xi}_1/M$ for $i = 0,1,\ldots,M$ and let $\xi_2(J,i) = \underline{\xi}_2(J) + i(\bar{\xi}_2(J) - \underline{\xi}_2(J))/M$, $i = 0,\ldots,M$. Furthermore, let $\tilde{\xi}_2(J,i) = (\xi_2(J,i) + \xi_2(J,i-1))/2$, $i = 1,\ldots,M$, and let J^* = Integer part of $(\lambda + 4\sqrt{\lambda})$. Then, a numerical approximation to the relative efficiency of $\hat{\sigma}^2_{BI}$ is given by

$$(3.8)\quad RE(\hat{\sigma}^2_{BI};\zeta,\frac{\psi}{\sigma^2},M,\nu) \cong \frac{2}{n+1}\left\{\sum_{j=0}^{J^*} p(j|\lambda)\cdot\right.$$

$$\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\left[\frac{1}{n}\left(\frac{\psi}{\sigma^2}+2\xi_1(i_1)\right)\cdot\frac{M_1\left(\dfrac{\frac{\psi}{\sigma^2}+2\tilde{\xi}_1(i_1)}{2n\varkappa^2},\frac{n}{2}+\nu-1,2\tilde{\xi}_2(j,i_2)/n\varkappa^2\right)}{M_1\left(\left(\frac{\psi}{\sigma^2}+2\tilde{\xi}_1(i_1)\right)/2n\varkappa^2,\frac{n}{2}+\nu,2\tilde{\xi}_2(j,i_2)/n\varkappa^2\right)}\right.$$

$$\left.-1\right]^2\cdot\left(G(\xi_1(i_1)|\tfrac{n-1}{2}) - G(\xi_1(i_1-1)|\tfrac{n-1}{2})\right)\cdot$$

$$\left.\left(G(\xi_2(j,i_2)|\tfrac{1}{2}+j) - G(\xi_2(j,i_2-1)|\tfrac{1}{2}+j)\right)\right\}^{-1}.$$

The functions $G(x|\frac{1}{2}+j)$, $j = 0,1,\ldots$ can be computed recursively according to the formula

$$(3.9)\qquad G(x|\tfrac{1}{2}+j) = \begin{cases} -\dfrac{1}{\Gamma(\frac{1}{2}+j)}x^{j-\frac{1}{2}}e^{-x} + G(x|j-\tfrac{1}{2}), & j \geq 1 \\ 2\Phi(\sqrt{2x}) - 1, & j = 0 \end{cases}.$$

The function $G(x|\frac{n-1}{2})$ is computed similarly if n is even. If n is odd we apply a similar recursion with $G(x|1) = 1-e^{-x}$. In Table 1 we provide values of the RE functions of $\hat{\sigma}^2_{BD}$ and $\hat{\sigma}^2_{BI}$ for $n = 10$, $\frac{\psi}{\sigma^2} = 2,6,10$; $\nu = 2$, $\kappa = 2$ and $\zeta = 0$, $.5$ and 1. We see that for $\zeta = 0$ $\hat{\sigma}^2_{BI}$ is more efficient than $\hat{\sigma}^2_{BD}$. Furthermore, for $\zeta = 0$ and $\frac{\psi}{\sigma^2}$ small $\hat{\sigma}^2_{BI}$ considerably more efficient than the best equivariant estimator $\hat{\sigma}^2_E$. However, when $\zeta \geq .5$ $\hat{\sigma}^2_{BD}$ is generally more efficient than $\hat{\sigma}^2_{BI}$, but both estimators may be less efficient than $\hat{\sigma}^2_E$ (recall that $\hat{\sigma}^2_E$ is minimax!).

Table 1. Relative Efficiency Values of $\hat{\sigma}^2_{BD}$ and $\hat{\sigma}^2_{BI}$ for samples of size $n = 10$.

ψ/σ^2	ν	κ	ζ	$RE(\hat{\sigma}^2_{BD})$	$RE(\hat{\sigma}^2_{BI})$
2.0	2.0	2.0	0.0	1.378	1.640
6.0	2.0	2.0	0.0	0.963	3.387
10.0	2.0	2.0	0.0	0.339	0.969
2.0	2.0	2.0	0.5	1.351	0.467
6.0	2.0	2.0	0.5	0.933	0.965
10.0	2.0	2.0	0.5	0.331	0.276
2.0	2.0	2.0	1.0	1.275	0.072
6.0	2.0	2.0	1.0	0.353	0.149
10.0	2.0	2.0	1.0	0.360	0.043

4. Estimating the Variance of a Finite Population

Let $x_1,\ldots,x_N$ be the values of N units in a finite population. We consider the problem of estimating the variance $\sigma^2_N = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$, $\mu = \frac{1}{N}\sum_{i=1}^{N}x_i$ on the basis of a sample of n values $X_1,\ldots,X_n$ chosen from that population. Zacks and Solomon [4] presented the form of Bayes estimators of σ^2_N. We derive here the Bayes estimator for the squared-error loss

when the model is that $x_1,\ldots,x_N$ are conditionally i.i.d. $N(\mu,\sigma^2)$ and that $\mu \sim N(\bar{\mu},D^2)$, $1/2\sigma^2 \sim \text{Gamma}(\psi,\nu)$. This model actually implies that the variates in the population are exchangeable random variables having a distribution which is a mixtrue of normal distributions. Without loss of generality one can assume that the sample consists of the first n variates $x_1,\ldots,x_n$. Let $\bar{x}_n$ be the sample mean and $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2$ the sample (classical) estimate of σ_N^2. Let $\bar{x}^*_{N-n}$ be the mean of the population variates which are not in the sample and $\tau^2_{N-n} = \frac{1}{N-n}\sum_{i=n+1}^{N}(x_i-\bar{x}^*_{N-n})^2$. It is shown in [4] that

$$\sigma_N^2 = \frac{n}{N}\hat{\sigma}_n^2 + (1-\frac{n}{N})\tau^2_{N-n} + \frac{n}{N}(1-\frac{n}{N})(\bar{x}_n - \bar{x}^*_{N-n})^2 . \tag{4.1}$$

We derive here the Bayes estimator $\hat{\sigma}_B^2 = E\{\sigma_N^2 | \underset{\sim}{x}_n\}$ according to the above model. One should determine the posterior expectations of τ^2_{N-n} and of $(\bar{x}_n - \bar{x}^*_{N-n})^2$, given the sample values $\underset{\sim}{x}_n = (x_1,\ldots,x_n)$. Notice first that since $x_1,\ldots,x_N$ are conditionally i.i.d, given μ,σ^2

$$\begin{aligned} &E\{\tau^2_{N-n} | \underset{\sim}{x}_n, \sigma^2\} = \frac{N-n-1}{N-n}\,\sigma^2 \\ &E\{(\bar{x}_n-\bar{x}^*_{N-n})^2 | \underset{\sim}{x}_n, \sigma^2, \mu\} = \frac{\sigma^2}{N-n} + (\mu-\bar{x}_n)^2 . \end{aligned} \tag{4.2}$$

Hence,

$$\begin{aligned} \hat{\sigma}_B^2 &= E\{\sigma_N^2 \mid \underset{\sim}{x}_n\} = \\ &\frac{n}{N}\,\hat{\sigma}_n^2 + (1-\frac{n}{N})(1-\frac{1}{N})E\{\sigma^2 | \underset{\sim}{x}_n\} + \frac{n}{N}(1-\frac{n}{N})E\{\mu-\bar{x}_n)^2 | \underset{\sim}{x}_n\} . \end{aligned} \tag{4.3}$$

We have seen in Section 2 that

$$\text{(4.4)}\qquad E\{\sigma^2 \mid \underset{\sim}{x}_n\} = E\{\sigma^2 \mid \bar{x}_n, \hat{\sigma}_n^2\}$$

$$= \frac{n\hat{\sigma}_n^2 + \psi}{n+2\nu-1} \cdot \frac{M_1(\lambda, \frac{n}{2}+\nu-1, \delta^2)}{M_1(\lambda, \frac{n}{2}+\nu, \delta^2)} ,$$

where $\lambda = (n\hat{\sigma}_n^2 + \psi)/2nD^2$ and $\delta^2 = (\bar{x} - \bar{\mu})^2/D^2$. To derive the posterior expectation of $(\mu - \bar{x}_n)^2$ we write first (see Zellner [5; pp. 22])

$$\text{(4.5)}\qquad E\{(\mu-\bar{x}_n)^2 \mid \sigma^2, \bar{x}_n, \hat{\sigma}_n^2\} = \frac{\sigma^2}{n} W + (\bar{x}_n - \bar{\mu})^2(1-W)^2,$$

where $W = D^2/(D^2 + \sigma^2/n)$. Finally, since $\frac{\sigma^2}{n} W = D^2(1 + 2n\theta D^2)^{-1}$ and $(1-W)^2 = (1 + 2n\theta D^2)^{-2}$ we obtain

$$\text{(4.6)}\qquad E\{(\mu-\bar{x}_n)^2 \mid \bar{x}_n, \hat{\sigma}_n^2\} = D^2 E\{(1+2n\theta D^2)^{-1} \mid \bar{x}_n, \hat{\sigma}_n^2\}$$

$$+ (\bar{x}_n - \bar{\mu})^2 E\{(1+2n\theta D^2)^{-2} \mid \bar{x}_n, \hat{\sigma}_n^2\}$$

and the Bayes estimator of σ_N^2 is

$$\text{(4.7)}\qquad \hat{\sigma}_B^2 = \frac{n}{N}\hat{\sigma}_n^2 + (1-\frac{n}{N})(1-\frac{1}{N}) \frac{n\hat{\sigma}_n^2 + \psi}{n+2\nu-2} \cdot \frac{M_1(\lambda, \frac{n}{2}+\nu-1, \delta^2)}{M_1(\lambda, \frac{n}{2}+\nu, \delta^2)}$$

$$+ \frac{n}{N}(1-\frac{n}{N})D^2 \; \frac{M_3(\lambda, \frac{n}{2}+\nu, \delta^2) + \delta^2 M_5(\lambda, \frac{n}{2}+\nu, \delta^2)}{M_1(\lambda, \frac{n}{2}+\nu, \delta^2)} .$$

Prior risk comparisons of the estimator $\hat{\sigma}_B^2$ with the classical estimator $\hat{\sigma}_n^2$ are given in [4].

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Appendix: FORTRAN Program for the Computation of $M_r(\lambda, \nu, \delta^2)$

The subroutine function is called BM(IR,AL,V,DS) where IR ← r, AL ← λ, v ← ν, DS ← δ^2. The subroutine function consists of three main parts (lines 10-170; 180-430 and 440-680). This is then supported by FUNCTION C(l,K) (lines 690-850) which computed the combinatorial function $\binom{K}{L}$. FUNCTION POS(J,AL) (lines 860-1070) computes the Poisson c.d.f. with mean AL ← λ at J ← j. Finally, FUNCTION DNDX(X) (lines 1080-1270) computes the standard normal integral $\Phi(x)$. In addition, the gamma function GAMF(W) ← Γ(W) is utilized in line 300. This function was computed with the computer library subroutine. If such a routine is not available one should supplement a subroutine FUNCTION GAMF(W).

```
00010       FUNCTION BM(IR,AL,V,DS)
00020       JR=IR
00030       BL=AL
00040       W=V
00050       ES=DS/2.
00060       IW=INT(W)
00070       K=(JR-1)/2
00080       AS=ES+4.*SQRT(ES)
00090       JS=INT(AS)+1
00100       BM=0.
00110       DO 1 J=1,JS
00120       JJ=J-1
00130       PJ=POS(J,ES)-POS(JJ,ES)
00140       BM=BM+PJ*R(JJ,K,BL,W,ES)
00150     1 CONTINUE
00160       RETURN
00170       END
00180       FUNCTION R(J,K,AL,V,ES)
00190       I=J
00200       L=K
00210       BL=AL
00220       W=V
00230       IW=INT(W)
00240       IS=IW+I
00250       GS=ES
00260       CO=1.
00270       DO 2 M=1,I
00280       CO=-CO
00290     2 CONTINUE
00300       H=EXP(BL+GS)*(BL**IW)/GAMF(W)
```

```
      H=H*CO
      CO=-1.
      T=0.
      DO 1 I=1,IS
      II=I-1
      CO=-CO
      IL=IW-L-II
      FI=CO*C(II,IS-1)*E(IL,BL)
      T=T+FI
    1 CONTINUE
      R=T*H
      RETURN
      END
      FUNCTION E(I,AL)
      L=I
      BL=AL
      PHI=3.1415927
      Z=SQRT(2.*BL)
      A0=2.*SQRT(PHI/BL)*(1.-CNDX(Z))
      IF(L) 3,1,2
    1 E=A0
      GO TO 10
    2 B0=A0
      DO 5 K=1,L
      AK=K
      B0=(EXP(-BL)+(AK-.5)*B0)/BL
    5 CONTINUE
      E=B0
      GO TO 10
    3 B0=A0
      M=-L
      DO 6 K=1,M
      AK=K
      B0=(EXP(-BL)-BL*B0)/(AK-.5)
    6 CONTINUE
      E=B0
   10 RETURN
      END
      FUNCTION C(L,K)
      I=L
      J=K
      IF(J-I) 1,2,3
    1 C=0.
      GO TO 10
    2 C=1.
      GO TO 10
    3 IF(I) 1,2,4
    4 C=1.
      DO 5 M=1,I
      BL=M
      DL=J-M+1
      C=C*DL/BL
    5 CONTINUE
   10 RETURN
      END
```

```
      FUNCTION POS(J,AL)
      I=J
      B=AL
      IF(B.GE.10.) GO TO 8
      IF(I) 1,2,3
    1 POS=0.
      GO TO 10
    2 POS=EXP(-B)
      GO TO 10
    3 POS=EXP(-B)
      F=POS
      DO 4 K=1,I
      AK=K
      F=F*B/AK
      POS=POS+F
    4 CONTINUE
      GO TO 10
    8 AI=I+.5
      ZI=(AI-B)/SQRT(B)
      POS=CNDX(ZI)
   10 RETURN
      END
      FUNCTION CNDX(X)
      Y=X
      ISWTCH=0
      IF(Y) 1,2,2
    1 Y=ABS(Y)
      ISWTCH=1
    2 P=.2316419
      B1=.31938153
      B2=-.35656378
      B3=1.7814779
      B4=-1.8212559
      B5=1.3302744
      T=1./(1.+P*Y)
      R=.3989423*EXP(-Y*Y/2.)
      QNDX=1.-R*(B1*T+B2*T*T+B3*T*T*T+B4*(T**4)+B5*(T**5))
      IF(ISWTCH) 3,4,3
    3 QNDX=1.-QNDX
    4 CNDX=QNDX
      RETURN
      END
```

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A Bayes estimator of the variance of a normal distribution $N(\mu, \sigma^2)$, when μ is unknown, is developed for squared-error loss and conjugate priors of independent parameters. ~~The estimator was suggested in Zacks [3; pp. 366].~~ In the present study its formula is developed and its relative efficiency is compared with that of the Bayes estimator ~~given in DeGroot [2]~~ and with that of the best equivariant estimator. Application of the estimator to the estimation of the variance of a finite population is provided.

mu